Technical Comment

Comment on "The Principal Minor Test for Semidefinite Matrices"

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PRUSSING[1] points out the dangers of trying to use the principal minor test for positive definite matrices as a test for a positive semidefinite matrix. Kerr[2] agrees, but feels that modifying the principal minor test to suit the new task is too expensive computationally. He recommends using the singular value decomposition (SVD) of the matrix as a more efficient and more stable technique to evaluate definiteness. A Choleski factorization, although computationally faster, is rejected because it is not stable for a semidefinite matrix. As an indicator of computational speed, the flop count for a Choleski factorization is given[3] as $n^3/6$, whereas the flop count for an SVD is given as $2n^3/3$.

This Comment suggests a different concept for this test, namely, the inertia of a matrix. The computational burden is similar to that of a Choleski factorization and software to perform this calculation is available in LINPACK.[4]

The inertia of a symmetric matrix is defined as the number triple $\{\pi, \nu, \zeta\}$, where π is the number of positive eigenvalues, ν is the number of negative eigenvalues, and ζ is the number of zero eigenvalues. For an nth-order matrix, the statement that it is positive definite is equivalent to the statement that its inertia is $\{n,0,0\}$. Similarly, a positive semidefinite matrix would have an inertia of $\{m,0,p\}$, $m+p=n$. The flop count for the matrix inertia calculation is $n^3/6$, nearly four times faster than the equivalent SVD. Although the matrix inertia calculation does not indicate the magnitude of the matrix eigenvalues, it can be used to discover eigenvalue bounds (see "Slicing the Spectrum" in Ref. 5 for a lucid commentary).

In summary, using LINPACK software to calculate the inertia of a matrix is a computationally efficient way to check whether the matrix is positive definite, positive semidefinite, or indefinite. The SVD is more suitable if information on why the matrix is not positive definite is required.

References

[1]Prussing, J. E., "The Principal Minor Test for Semidefinite Matrices," *Journal of Guidance, Control, and Dynamics,* Vol. 9, Jan.-Feb. 1986, pp. 121–122.

[2]Kerr, T. H., "Testing Matrices for Definiteness and Application Examples that Spawn the Need," *Journal of Guidance, Control, and Dynamics,* Vol. 10, Sept.-Oct. 1987, pp. 503–506.

[3]Golub, G. H. and Van Loan, C. F., *Matrix Computations,* John Hopkins Univ. Press, Baltimore, MD, 1983, pp. 89, 175.

[4]Dongarra, J. J., et al., *LINPACK User's Guide,* Society for Industrial and Applied Mathematics, Philadelphia, PA, 1979, Chap. 5.

[5]Parlett, B. N., *The Symmetric Eigenvalue Problem,* Prentice-Hall, Englewood Cliffs, NJ, 1980, p. 46.

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Reply by Author to J. L. Tietze

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I welcome Tietze's interest in this problem and his intriguing suggestion to use an existing LINPACK software implementation of an algorithm that will reveal the "definiteness" or "semidefiniteness" of a candidate symmetric matrix based on computing its inertia. However, the computational burden of this new approach is identical to that of a version of singular value decomposition (SVD) that is specially tailored to exploit the symmetry of the matrices under test, as discussed further.

In my prior Engineering Note,[2] I agreed with the observations of Ref. 1 regarding the dangers of trying to reveal positive semidefiniteness of a matrix through the use of the popular slight variation of the standard "principal minor test." In Ref. 2, I also emphasized the theoretically correct version of this same test for checking positive semidefinite matrices. However, reservations were expressed that even this correctly modified version is both too numerically ill-behaved and too expensive to be a practical and useful test for the matrices of higher dimensions that are typically encountered outside the class room, and so a tractable computational test based on SVD was recommended. I have, subsequently, learned that it was at least as early as 1974 in Ref. 4 that SVD was first observed to be "the only reliable method for establishing the rank of a matrix."

Use of the Choleski factorization as the basis of a practical computational test was rejected in Ref. 2 because it is not numerically stable enough to serve as a valid test for semidefinite matrices. An example of its failure in the role of a test was cited from Ref. 3, p. 90, example 5.2-2, despite the matrix under test even being positive definite.

It is agreed that the operations count for a Choleski factorization is $n^3/6$, whereas that of a general SVD is $2n^3/3$. However, there is an SVD refinement known as Aasen's method (Ref. 3, pp. 101–106) that takes advantage of the symmetry of the matrices being examined and has an associated "flop" count of only $n^3/6$, no more than that of the proposed technique based on determining a matrix's inertia. However, as Tietze observed, determining the inertia of a matrix does not reveal the magnitude of the underlying eigenvalues, whereas Aasen's method does so at no additional computational expense or delay.

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References

[1]Prussing, J. E., "The Principal Minor Test for Semidefinite Matrices," *Journal of Guidance, Control, and Dynamics,* Vol. 9, Jan–Feb. 1986, pp. 121–122.

[2]Kerr, T. H., "Testing Matrices for Positive Definiteness and Semidefiniteness and Application Examples that Spawn the Need," *Journal of Guidance, Control, and Dynamics,* Vol. 10, Sept.-Oct. 1987, pp. 503–506.

[3]Golub, G. H. and Van Loan, C. F., *Matrix Computations,* Johns Hopkins Univ. Press, Baltimore, MD, 1983.

[4]Lawson, C. L. and Hanson, R. J., *Solving Least Squares Problems,* Prentice-Hall, Englewood Cliffs, NJ, 1974.

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